ISRAEL JOURNAL OF MATHEMATICS, Vol. 22, No. 1, 1975

ALGEBRAIC PROPERTIES OF NUMBER THEORIES

BY

A. MACINTYRE AND H. SIMMONS

ABSTRACT

Among other things we prove the following. (A) A number theory is convex if and only if it is inductive. (B) No r.e. number theory has JEP. (C) No number theory has AP. We also give some information about the hard cores of number theories.

0. Introduction

This paper is a continuation of [0]. In it we expand on the remarks made in [0, Postscript], answer some of the questions of [0, Section 4], and give some new results.

In Section 1 we give a lemma which is used several times throughout the paper, and deduce from it the above result (A). In Section 2 we give the details of the improvements mentioned in [0, Postscript].

In Section 3 we consider various aspects of JEP for number theories; in particular we prove (B). In Section 4 we turn to AP for number theories, and prove (C). In Section 5 we look at hard cores of number theories and their connection with finite forcing generic structures.

In the last section, Section 6, we make some further remarks and ask some questions.

The results given here have been obtained by one or other of us since the time of writing [0]. At that time we had seen only a summary of [3], although later we saw the full version. A lot of what follows owes a great deal to [3]. It should be noted, however, that almost all the results of [3] are proved for full number theory N only. Here we are concerned with all number theories.

We use the same notation and terminology of [0], in particular we use the notations $\mathscr{E}$, $\mathscr{E}_T$, $\mathscr{F}_T$, $\mathscr{S}(T)$, $\mathscr{M}(T)$, B, P, N, $\mathfrak{N}$, $\exists_{n+1}$, $\forall_{n+1}$ and the terminology "submodel", "number structure", "number theory". Any new notation or terminology is indicated in the text.

Received October 21, 1974

We use the references of [0] together with some new items (i.e. [19]–[24]), and we have taken this opportunity to give new information for some items of [0] (i.e. [3], [7], [13], [16], [17], [18]). We also take this opportunity to give the following corrections to [0].

(1) p. 328, l. 18:—In the definition of $K(v,x)$, "$\vee$" should be "$\wedge$".

(2) p. 330, l. 13:—$\cdots$ formula $d^{(n)}(v,w)\cdots$.

(3) p. 333, l. 11:—$\cdots\mathfrak{A}\in\mathscr{C}$.

(4) p. 334, l. 7:—Both ocurrences of "ϕ" should be "θ".

(5) p. 336. l. 1:—$\cdots$cardinal κ $\cdots$.

(6) p. 336, l. 11:—$\cdots$ $i\,(\mathscr{F}_{\mathrm{T}})=\cdots$.

1. Intersection of number structures

Because peano number theory P has induction axioms it also has (internally) definable skolem functions. If we use these skolem functions with a little care we obtain the following lemma.

LEMMA 1.1. *For each $n\in\omega$ and each $\exists_{n+1}$-formula $\theta(v,w)$ there is an $\exists_{n+1}$-formula $\mu(v,w)$ such that the sentences*

(i) $(\forall w)[(\exists v)\theta(v,w)\rightarrow(\exists v)\mu(v,w)]$,

(ii) $(\forall\, w,v,v')[\mu(v,w)\wedge\mu(v',w)\rightarrow v=v']$,

(iii) $(\forall w,v)[\mu(v,w)\rightarrow\theta(v,w)]$,

are provable in P.

Notice that here v, w need not be single variables but can be finite strings of variables. Notice also that for $n=0$ the three sentences (i, ii, iii) are $\forall_2$, $\forall_1$ and $\forall_2$, hence are provable in B.

Let us use this lemma to prove the following theorem.

THEOREM 1.2. *Let $\{\mathfrak{B}_i: i\in I\}$ be a family of number structures and $\mathfrak{A}$ a number structure such that for each $i\in I$, $\mathfrak{B}_i\subseteq\mathfrak{A}$. Let*

$$\mathfrak{C}=\bigcap\{\mathfrak{B}_i: i\in I\}.$$

Let $\phi(w_1,\cdots,w_s)$ be an $\exists_1$-formula and let $c_1,\cdots,c_s$ be elements of $\mathfrak{C}$ such that for each $i\in I$, $B_i\models\phi(c_1,\cdots,c_s)$. Then $\mathfrak{C}\models\phi(c_1,\cdots,c_s)$.

PROOF. Let

$$\phi(w_1,\ \cdots,w_s)=(\exists v_1,\cdots,v_r)\theta(v_1,\cdots,v_r,w_1,\cdots,w_s),$$

where θ is a quantifier-free formula, and let $\mu(v_1,\cdots,v_r,w_1,\cdots,w_s)$ be the corresponding $\exists_1$-formula given by Lemma 1.1.

Let $i \in I$. Part (i) of the lemma shows that there are elements $b_1, \cdots, b_r$ of $\mathfrak{B}_i$ such that

$$\mathfrak{B}_i \models \mu(b_1, \cdots, b_r, c_1, \cdots, c_s)$$

and so (since μ is $\exists_1$)

$$\mathfrak{A} \models \mu(b_1, \cdots, b_r, c_1, \cdots, c_s).$$

Part (ii) of the lemma now shows that the $b_1, \cdots, b_r$ are independent of i and so are elements of $\mathfrak{C}$. Part (iii) of the lemma gives

$$\mathfrak{A} \models \theta(b_1, \cdots, b_r, c_1, \cdots, c_s)$$

so that (since θ is quantifier-free)

$$\mathfrak{C} \models \theta(b_1, \cdots, b_r, c_1, \cdots, c_s),$$

which gives the required result.

COROLLARY 1.3. *Let $\mathfrak{A}$ be a number structure and $\{\mathfrak{B}_i : i \in I\}$ a family of structures such that for each $i \in I$ $\mathfrak{B}_i <_1 \mathfrak{A}$. Let*

$$\mathfrak{C} = \cap \{\mathfrak{B}_i : i \in I\}.$$

Then $\mathfrak{C} <_1 \mathfrak{A}$.

Remember that a theory T is convex if for each model $\mathfrak{A}$ of T and family $\{\mathfrak{B}_i : i \in I\}$ of substructures of $\mathfrak{A}$, each of which is also a model of T, the intersection

$$\mathfrak{C} = \cap \{\mathfrak{B}_i : i \in I\}$$

is again a model of T (provided the intersection is non-empty). There are syntactical characterizations of convex theories, in particular each convex theory is $\forall_2$-axiomatizable. In general the converse of this is false, but for number theories it is true.

THEOREM 1.4. *A number theory is convex if and only if it is $\forall_2$-axiomatizable.*

PROOF. Let T be any number theory. Let $\{\mathfrak{B}_i : i \in I\}$ be a family of models of T and let $\mathfrak{A}$ be a model of T such that for each $i \in I$, $\mathfrak{B}_i \subseteq \mathfrak{A}$. Let

$$\mathfrak{C} = \cap \{\mathfrak{B}_i : i \in I\}.$$

We show that $\mathfrak{C} \models \mathrm{T} \cap \forall_2$.

Consider any $\sigma \in \mathrm{T} \cap \forall_2$ and let $\sigma = (\forall w_1, \cdots, w_s)\phi(w_1, \cdots, w_s)$, where ϕ is an $\exists_1$-formula. Consider any elements $c_1, \cdots, c_s$ of $\mathfrak{C}$. For each $i \in I$, $\mathfrak{B}_i \models \sigma$, so that $\mathfrak{B}_i \models \phi(c_1, \cdots, c_s)$. Theorem 1.2 now gives $\mathfrak{C} \models \phi(c_1, \cdots, c_s)$, so that (since $c_1, \cdots, c_s$ are arbitrary elements of $\mathfrak{C}$) $\mathfrak{C} \models \sigma$.

2. Definable standard parts

The central result of [0] is the existence of a certain formula I which defines in each number structure $\mathfrak{A} \in \mathscr{E}$ the standard part ω of $\mathfrak{A}$. This formula I is $\exists_3$ and is constructed using a certain creative set. In [3, Theorem 1.32] Hirschfeld proves a similar result for number structures $\mathfrak{A} \in \mathscr{E}_{\mathrm{N}}$. Starting from any simple set Hirschfeld obtains an $\exists_2$-definition of ω. In this section we show that by using a particular simple set Hirschfeld's method works for all number structures $\mathfrak{A} \in \mathscr{E}$. This leads to the improvements mentioned in the postscript of [0].

Our proof is simply a careful analysis of Hirschfeld's proof. Hirschfeld uses the truth of certain number theoretic sentences. Here we must check that, as well as being true, these sentences are also provable in B.

As always we need a formal version of the enumeration theorem.

(2.1) *There is a certain $\exists_1$-formula $d(v, w)$ such that for each $\exists_1$-formula $\theta(v)$ there is some $t \in \omega$ such that* $\mathrm{B} \models (\forall v)[\theta(v) \leftrightarrow d(v, t)]$.

Note that here v, w are single variables, not finite sequences of variables.

We need a description of a certain simple set. The simple set we use is the original one as constructed by Post, so we formalize [14, p. 106, Theorem II].

Let $\theta(v, w)$ be the $\exists_1$-formula $d(v, w) \wedge 2w < v$, and let $\mu(v, w)$ be the $\exists_1$-formula corresponding to θ given by Lemma 1.1. Let α, β, γ be three sentences displayed and (i, ii, iii) of Lemma 1.1. Since each of α, β, γ is an $\forall_2$-sentence, we have the following.

(2.2) *Each of the three sentences α, β, γ is provable in* B.

Now let $S(v)$ be the $\exists_1$-formula $(\exists w)\mu(v, w)$, so $S(v)$ describes the original simple set of Post. Notice that the simplicity of S is actually provable.

LEMMA 2.1. *The sentence*

$$(\forall w)[(\forall v)[d(v, w) \rightarrow \neg S(v)] \rightarrow (\forall v)[d(v, w) \rightarrow v \leqq 2w]]$$

is provable in B.

PROOF. We demonstrate the contrapositive sentence. Using first α and then γ we see that the implications

$$\begin{aligned}(\exists v)[d(v,w) \wedge 2w < v] &\rightarrow (\exists v)\mu(v,w)\\ &\rightarrow (\exists v)[\mu(v,w) \wedge d(v,w)]\\ &\rightarrow (\exists v)[S(v) \wedge d(v,w)]\end{aligned}$$

hold in B, as required.

Lemma 2.1 gives us the following crucial theorem (c.f. [3, Lemma 1.31]).

THEOREM 2.2. *For each structure* $\mathfrak{A} \in \mathscr{E}$ *and element* a *of* $\mathfrak{A}$, *if* $\mathfrak{A} \models \neg S(a)$ *then* $a \in \omega$ (*i.e.* a *is a standard element of* $\mathfrak{A}$).

PROOF. Suppose $\mathfrak{A} \models \neg S(a)$ for such a structure $\mathfrak{A}$ and element a. Since $\neg S(v) \in \forall_1$, [15, Theorem 2.1] gives us some $\theta(v) \in \exists_1$ such that

$$\mathfrak{A} \models \theta(a), \quad \mathfrak{A} \models (\forall v)[\theta(v) \rightarrow \neg S(v)].$$

Now (2.1) shows that we may assume that $\theta(v) = d(v,t)$, for some $t \in \omega$, hence using Lemma 2.1 we have

$$\mathfrak{A} \models d(a,t), \quad \mathfrak{A} \models (\forall v)[d(v,t) \rightarrow v \leqq 2t].$$

Thus $a \leqq 2t$ and so $a \in \omega$, as required.

Let $I(x)$ be the $\exists_2$-formula

$$(\exists v)[x \leqq v \leqq 2x \wedge \neg S(v)].$$

Theorem 2.2 shows that for each $\mathfrak{A} \in \mathscr{E}$, $I^{\mathfrak{A}} \subseteq \omega$. To show the converse inclusion we use the following.

LEMMA 2.3. (i) *For each* $n \in \omega$, $\mathrm{B} \vdash I(n)$.

(ii) $\mathrm{P} \vdash (\forall x) I(x)$.

PROOF. (i) Inside B we have

$$\begin{aligned}\neg I(n) &\rightarrow \wedge\{S(n+i):\ 0 \leqq i \leqq n)\\ &\rightarrow (\exists w_0, \cdots, w_n) \wedge \{\mu(n+i, w_i):\ 0 \leqq i \leqq n\}\\ &\rightarrow (\exists w_0, \cdots, w_n) \wedge \{w_i < n \wedge \mu(n+i, w_i): 0 \leqq i \leqq n\},\end{aligned}$$

where the last implication holds since $\gamma \in \mathrm{B}$. The pigeon-hole principle (in the real world) now gives (in B)

$$\neg I(n) \rightarrow (\exists w, v_1, v_2)[v_1 \neq v_2 \wedge \mu(v_1, w) \wedge \mu(v_2, w)],$$

so the required result follows using β.

(ii) This follows by formalizing the proof of (i) inside P. Notice that this requires a formal version of the pigeon-hole principle.

We have now improved [0, Theorem 2.1] to the following.

THEOREM 2.4. *There is a certain $\exists_2$-formula I containing just one free variable such for each $\mathfrak{A} \in \mathscr{E}$, $I^{\mathfrak{A}} = \omega$.*

We will now improve [0, Theorem 2.5], i.e. we will prove the following.

THEOREM 2.5. *For each $n \in \omega$ there is an $\forall_{n+3}$-sentence $\rho^{(n)}$ such that $\rho^{(n)} \in \mathrm{P}$ and for each $\forall_{n+2}$-axiomatizable theory $\mathrm{T} \supseteq \mathrm{B}$, $\rho^{(n)} \notin \mathrm{T}$.*

We have already proved this for the case $n = 0$, we simply put $\rho^{(0)} = (\forall x)I(x)$. In [0] we explained how to obtain the general version of the theorem from this particular case, i.e. we simply relativize all of the preceding work. However, this explanation is misleading since such a simple-minded process gives us the weaker version of Theorem 2.5 obtained by replacing "B" by "$\mathrm{P} \cap \forall_{n+2}$". To obtain Theorem 2.5 by relativization we must note where (2.1) and (2.2) were used in the preceding work, and find suitable generalized versions of them which are weaker than the simple-minded relativized versions.

Let n be some fixed natural number.

Making use of recursive pairing functions of B, we have the following analogue of (2.1).

(2.3) *There is a certain $\exists_{n+1}$-formula $d^{(n)}(v, w)$ such that for each $\exists_{n+1}$-formula $\theta(v)$ there is some $t \in \omega$ such that $\mathrm{B} \vdash (\forall v)[\theta(v) \leftrightarrow d^{(n)}(v, t)]$.*

Let $\theta^{(n)}(v, w)$ be the $\exists_{n+1}$-formula $d^{(n)}(v, w) \wedge 2w < v$, and let $\mu^{(n)}(v, w)$ be the $\exists_{n+1}$-formula corresponding to $\theta^{(n)}$ given by Lemma 1.1. Let $\alpha^{(n)}$, $\beta^{(n)}$, $\gamma^{(n)}$ be the three corresponding sentences of Lemma 1.1. It turns out that we do not need an analogue of (2.2). All we need to know is the following.

(2.4) *Each of $\alpha^{(n)}$, $\beta^{(n)}$, $\gamma^{(n)}$ is an $\forall_{n+2}$-sentence and $\mathrm{P} \vdash \alpha^{(n)} \wedge \beta^{(n)} \wedge \gamma^{(n)}$.*

Let $S^{(n)}(v)$ be the $\exists_{n+1}$-formula $(\exists w)\mu^{(n)}(v, w)$ and let $I^{(n)}(v)$ be the $\exists_{n+2}$-formula

$$(\exists v)[x \leqq v \leqq 2x \wedge \neg S^{(n)}(v)].$$

Notice that the initial quantifier of $I^{(n)}(x)$ is bounded, so that we have the following.

(2.5) *There is an* $\forall_{n+1}$*-formula* $J^{(n)}(x)$ *such that the* $\forall_{n+3}$ *sentence*

$$\eta^{(n)} = (\forall x)[I^{(n)}(x) \leftrightarrow J^{(n)}(x)]$$

is provable in P.

For each theory T let $\mathscr{E}_T^{(n)}$ be the class of structures $\mathfrak{A}$ such that there is some $\mathfrak{B} \models T$ with $\mathfrak{A} <_n \mathfrak{B}$, and for all $\mathfrak{B} \models T$,

$$\mathfrak{A} <_n \mathfrak{B} \Rightarrow \mathfrak{A} <_{n+1} \mathfrak{B}.$$

In particular $\mathscr{E}_T^{(0)} = \mathscr{E}_T$.

Working through Lemma 2.1, Theorem 2.2, Lemma 2.3 and making careful notes of how the sentences α, β, γ are used we obtain the following analogue of Theorem 2.4.

LEMMA 2.6. *For each theory* $T \supseteq B \cup \{\alpha^{(n)}, \beta^{(n)}, \gamma^{(n)}\}$ *and each* $\mathfrak{A} \in \mathscr{E}_T^{(n)}$, $I^{(n)\mathfrak{A}} = \omega$.

Now let $\rho^{(n)}$ be $\alpha^{(n)} \wedge \beta^{(n)} \wedge \gamma^{(n)} \wedge \eta^{(n)} \wedge (\forall x) J^{(n)}(x)$ so that $\rho^{(n)} \in \forall_{n+3}$.

PROOF OF THEOREM 2.5. Using the analogue of Lemma 2.3 (ii) and (2.4), (2.5), we see that $P \vdash \rho^{(n)}$.

Now consider any $\forall_{n+2}$-axiomatizable theory $T \supseteq B$, and suppose that $T \vdash \rho^{(n)}$. Let $\mathfrak{A}$, be any uncountable member of $\mathscr{E}_T^{(n)}$ having carrier set A.

Since T is $\forall_{n+2}$-axiomatizable, we have $\mathfrak{A} \models T$, so that $\mathfrak{A} \models \rho^{(n)}$. But then Lemma 2.6 with the sentence $\eta^{(n)}$ gives

$$\omega = I^{(n)\mathfrak{A}} = J^{(n)\mathfrak{A}} = A.$$

This is imposible (since A is uncountable) and so the theorem is proved.

3. The joint embedding property

A theory T is said to have JEP if for each two models $\mathfrak{A}$, $\mathfrak{B}$ of T there is a third model $\mathfrak{C}$ of T together with embeddings $\mathfrak{A} \to \mathfrak{C}$, $\mathfrak{B} \to \mathfrak{C}$. In particular any complete theory has JEP.

The following characterization is well known.

LEMMA 3.1. *A theory* T *has* JEP *if and only if for each pair* α, β *of* $\forall_1$*-sentences, if* $T \vdash \alpha \vee \beta$ *then* $T \vdash \alpha$ *or* $T \vdash \beta$.

Rabin has shown that no r.e. extension of P has JEP. Independently the first author noted that this result follows from the existence of a certain pair of recursively inseparable sets. This argument is incorporated in the proof of Theorem 3.4 (below). For some of the results of this and the next section we require something stronger than the existence of recursively inseparable sets, namely a version of Friedberg's splitting theorem. Friedberg's theorem can be stated as follows.

THEOREM 3.2. *For each r.e. set E there are r.e. sets L, R such that the following hold.*

(i) $E = L \cup R$.

(ii) $L \cap R = \varnothing$.

(iii) *For each pair of r.e. sets X, Y such that*

$$L \cap X = R \cap Y = \varnothing$$

there are r. e. sets X', Y' such that

$$X' - E = X - E,\quad Y' - E = Y - E$$

and both $X' \cap E$, $Y' \cap E$ are finite.

A statement of Friedberg's theorem in this form can be found in [20, Theorem 4]. Notice, however, that here we are not concerned with the incomparability of L, R and so we do not have a clause corresponding to 4.3 of [20, Theorem 4].

We will require the followihg consequence of (i, ii, iii) of Theorem 3.2.

(3.1) *For each pair of r.e. sets X, Y,*

$$L \cap X = \varnothing \Rightarrow X - R \text{ is r.e.}$$

$$R \cap Y = \varnothing \Rightarrow Y - L \text{ is r.e.}$$

As can be expected there is a certain amount of effectiveness and uniformity in the proof of Theorem 3.2. Analysing this proof we find that the following theorem holds. (It is not a good idea to analyse the proof given in [20] since the clause (4.3) intodoces a lot of unnecessary complications.)

THEOREM 3.3. *For each $\exists_1$-formula $E(v)$, there are $\exists_1$-formulas $L(v)$, $R(v)$ such that*

(i) $\mathrm{B} \vdash (\forall v)[E(v) \leftrightarrow L(v) \vee R(v)]$,

(ii) $\mathrm{B} \cap (\forall v)[\neg L(v) \vee \neg R(v)]$,

(iii) *condition* (3.1) *holds* (*where L, R are the r.e. sets defined by $L(v)$, $R(v)$*).

Notice that we do not state the obvious effective version of (3.1). We will say more about this in the next section.

Almost all of the results concerning number theories are based on diagonal arguments. Sometimes this diagonalization is done within the theory itself and sometimes it is done outside the theory (i.e. within the external recursive function theory). Sometimes an internal diagonal argument can be replaced by an external diagonal argument. This is always desirable since external arguments are easier to handle.

A case in point is the result [231, Lemma 3.3] of Don Jensen, which he proved using an internal diagonalization argument. We derive a more general version of [21, Lemma 3.3] from Theorem 3.3.

THEOREM 3.4. *Let γ be any true $\forall_1$--sentence (i.e. $\mathfrak{N} \models \gamma$). Let $\{T_i: i \in I\}$ be a set of number theories such that $\{T_i \cap \forall_1: i \in I\}$ is uniformly r.e. Then there are true $\forall_1$-sentences α, β such that the following hold.*

(i) $B \vdash \alpha \wedge \beta \rightarrow \gamma$.

(ii) $B \vdash \alpha \vee \beta$.

(iii) *For each $i \in I$ neither $T_i \vdash \alpha$ nor $T_i \vdash \beta$.*

PROOF. Let $\theta(v)$ be any $\exists_1$-formula which defines a non-recursive r.e. set E. We easily check that the formula $E(v) = \neg\gamma \vee \theta(v)$ also defines E. Let $L(v)$, $R(v)$ be the formulas given by Theorem 3.3, and let L, R be the corresponding r.e. sets. For each $n \in \omega$ we have

$$B \vdash \neg L(n) \wedge \neg R(n) \rightarrow \gamma$$

$$B \vdash \neg L(n) \vee \neg R(n).$$

Consider the two sets X, Y defined by

$$n \in X \Leftrightarrow \text{There is some } i \in I \text{ with } T_i \vdash \neg L(n)$$

$$n \in Y \Leftrightarrow \text{There is some } i \in I \text{ with } T_i \vdash \neg R(n).$$

These two sets are r.e. and clearly $L \cap X = R \cap Y = \varnothing$. Also Theorem 3.3 (ii) shows that $L \subseteq Y$, $R \subseteq X$, so that

$$X \cup Y - E = (X - R) \cup (Y - L).$$

This, with (3.1), shows that $X \cup Y - E$ is r.e.

Now E is not recursive and $E \subseteq X \cup Y$, so that $X \cup Y \neq \omega$. We can thus take any $n \notin X \cup Y$ and put $\alpha = \neg L(n)$, $\beta = \neg R(n)$.

Notice that the set E used in the proof is arbitrary and independent of the sentence γ, but this does not mean the sets L, R are independent of γ.

Remembering Lemma 3.1 we immediately obtain the following corollary.

COROLLARY 3.5. *Each number theory* T *with* $T \cap \forall_1$ *r.e. does not have* JEP.

Notice that there are complete number theories of complexity Δ_2^0.

Corollary 3.5 gives us some information about the number of f-generic structures for number theories. The following theorem verifies and extends the conjecture [0, §4, 8 (vi)]. Details can be found in [24].

THEOREM 3.6. *For each number theory* T, *if* $T \cap \forall_1$ *is r.e. then* $j(\mathscr{F}_T) = 2^{\aleph_0}$.

We now consider just how badly JEP can fail for a number theory. To do this we make use of a weak version of the elementary equivalence relation.

For each two structures $\mathfrak{A}, \mathfrak{B}$ we write $\mathfrak{A} \Rrightarrow (\exists_1)\, \mathfrak{B}$ if each $\exists_1$-sentence which holds in $\mathfrak{A}$ also holds in $\mathfrak{B}$. The following lemma is well known.

LEMMA 3.7. *For each two structures* $\mathfrak{A}, \mathfrak{B}$ *the following are equivalent.*

(i) $\mathfrak{A} \Rrightarrow (\exists_1)\, \mathfrak{B}$.

(ii) *There is a structure* $\mathfrak{C}$ *together with an embedding* $\mathfrak{A} \to \mathfrak{C}$ *and an elementary embedding* $\mathfrak{B} \to \mathfrak{C}$.

For each number theory T let $\mathscr{S}'(T)$ be the class $\mathscr{S}(T) \cap \mathscr{M}(B)$, and let $\mathscr{J}(T)$ be the class of number structures $\mathfrak{A}$ such that for each $\mathfrak{B} \models T$ there is some $\mathfrak{C} \models T$ together with embeddings $\mathfrak{A} \to \mathfrak{C}$, $\mathfrak{B} \to \mathfrak{C}$.

THEOREM 3.8. *For each number theory* T *the following hold.*

(i) $\mathscr{S}'(N) \subseteq \mathscr{J}(T) \subseteq \mathscr{S}'(T)$.

(ii) T *has* JEP *if and only if* $\mathscr{J}(T) = \mathscr{S}'(T)$.

PROOF. Suppose that $\mathfrak{A} \in \mathscr{S}'(N)$, so that $\mathfrak{A} \Rrightarrow (\exists_1)\, \mathfrak{N}$, and consider any $\mathfrak{B} \models T$. Since $\mathfrak{B}$ contains an isomorphic copy of $\mathfrak{N}$ (i.e. the standard part of $\mathfrak{B}$), we have $\mathfrak{N} \Rrightarrow (\exists_1)\, \mathfrak{B}$, so that $\mathfrak{A} \Rrightarrow (\exists_1)\, \mathfrak{B}$. Thus Lemma 3.7 shows that $\mathfrak{A} \in \mathscr{J}(T)$.

This proves the first inclusion of (i). The second inclusion and (ii) are trivial.

This theorem shows that $\mathscr{J}(T)$ is a measure of the joint embedding properties of T. We show that on this scale r.e. number theories fail to have JEP in the worst possible way.

THEOREM 3.9. *For each number theory* T, *if* $T \cap \forall 1$ *is r.e. then* $\mathscr{J}(T) = \mathscr{S}'(N)$.

PROOF. Let T be a number theory with $T \cap \forall_1$ r.e. and suppose $\mathfrak{C} \in \mathscr{J}(T)$. Let γ be any true $\forall_1$-sentence. We will show that $\mathfrak{C} \models \gamma$, so that $\mathfrak{A} \Rrightarrow (\exists_1)\, \mathfrak{N}$ and hence $\mathfrak{A} \in \mathscr{S}'(N)$.

By Theorem 3.4 there are (true) $\forall_1$-sentences α, β such that

(i) $B \vdash \alpha \wedge \beta \rightarrow \gamma$,

(ii) $B \vdash \alpha \vee \beta$,

(iii) $T \nvdash \alpha$ and $T \nvdash \beta$.

Let $\mathfrak{A}$, $\mathfrak{B}$ be models of T such that $\mathfrak{A} \models \neg\alpha$ and $\mathfrak{B} \models \neg\beta$. (These exist by (iii).) Since $\mathfrak{C} \in \mathscr{J}(T)$ there are models $\mathfrak{A}'$, $\mathfrak{B}'$ of T such that $\mathfrak{A}'$ is a common extension of $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{B}'$ is a common extension of $\mathfrak{B}$, $\mathfrak{C}$. Since $\neg\alpha, \neg\beta \in \exists_1$, we have $\mathfrak{A}' \models \neg\alpha$, $\mathfrak{B}' \models \neg\beta$, and so (ii) gives $\mathfrak{A}' \models \beta$, $\mathfrak{B}' \models \alpha$. But $\alpha, \beta \in \forall_1$, so that $\mathfrak{C} \models \alpha \wedge \beta$. Now (by definition of $\mathscr{J}(T)$) $\mathfrak{C} \models B$, and so (i) gives $\mathfrak{C} \models \gamma$, as required.

Finally, in this section, we give a necessary and sufficient condition for the joint embeddability of two number structures, We use the following observation.

LEMMA 3.10. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be structures such that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}$, $\mathfrak{A} \models B$, $\mathfrak{C} \models B$, and $\mathfrak{A}$ is cofinal in $\mathfrak{B}$. Then $\mathfrak{A} <_1 \mathfrak{B}$.*

PROOF. Let $\phi(v, w)$ be any quantifier free formula (where v, w are finite sequences of variables) and a any sequence of elements of $\mathfrak{A}$ such that $\mathfrak{A} \models (\forall v)\phi(v, a)$. Let u be a new variable and consider the formula $(\forall v \leqq u)\phi(v, w)$ (where $v \leqq u$ has the obvious meaning). Let $\theta(u, w)$ be any $\exists_1$-formula such that

$$B \vdash (\forall v \leqq u)\phi(v, w) \leftrightarrow \theta(u, w).$$

Consider any element b of $\mathfrak{B}$. Since $\mathfrak{A}$ is cofinal in $\mathfrak{B}$ there is some a' of $\mathfrak{A}$ such that $b \leqq a'$. Now $\mathfrak{A} \models (\forall v \leqq a')\phi(v, a)$, so that (by the above remarks) $\mathfrak{C} \models (\forall v \leqq a')\phi(v, a)$. In particular $\mathfrak{C} \models \phi(b, a)$, so that $\mathfrak{B} \models \phi(b, a)$.

This shows $\mathfrak{B} \models (\forall v)\phi(v, a)$, as required.

The next theorem should be compared with [21, Theorem 6(i)].

THEOREM 3.11. *For each two number structures $\mathfrak{A}_1$, $\mathfrak{A}_2$ the following are equivalent.*

(i) *$\mathfrak{A}_1$, $\mathfrak{A}_2$ have a common number structure extension.*

(ii) *$\mathfrak{A}_1 \Rrightarrow (\exists_1)\mathfrak{A}_2$ or $\mathfrak{A}_2 \Rrightarrow (\exists_1)\mathfrak{A}_1$.*

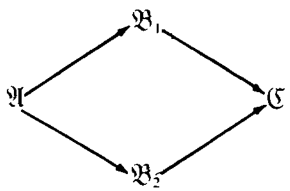
PROOF. (i) $\Rightarrow$ (ii). Suppose we have some $\mathfrak{C} \models B$ together with embeddings $\mathfrak{A}_i \rightarrow \mathfrak{C}$. Let $\mathfrak{B}_i$ be the initial section of $\mathfrak{C}$ generated by the image of $\mathfrak{A}_i$. Thus we have an embedding $\mathfrak{A}_i \rightarrow \mathfrak{B}_i$ which (by Lemma 3.10) is a 1-embedding.

Now $\mathfrak{C}$ is an end extension of both $\mathfrak{B}_1$, $\mathfrak{B}_2$ so that either $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ or $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$. This gives $\mathfrak{A}_1 \Rrightarrow (\exists_1)\mathfrak{A}_2$ or $\mathfrak{A}_2 \Rrightarrow (\exists_1)\mathfrak{A}_1$, respectively.

(ii) $\Rightarrow$ (i). This implication follows by Lemma 3.7.

4. The amalgamation property

A submodel $\mathfrak{A}$ of a theory T is an amalgamation base for T if for each pair of models $\mathfrak{B}_1$, $\mathfrak{B}_2$ of T and embeddings $\mathfrak{A} \to \mathfrak{B}_i$, there is a model $\mathfrak{C}$ of T and embeddings $\mathfrak{B}_i \to \mathfrak{C}$ such that the diagram



commutes. Let $\mathcal{A}(T)$ be the class of amalgamation bases for T. (We know that $\mathcal{A}(T)$ is exactly the class of pregeneric structures for T, but we make no use of this here.)

A theory T has AP if $\mathcal{M}(T) \subseteq \mathcal{A}(T)$. Notice also that a number theory T has JEP if and only if $\mathfrak{N} \in \mathcal{A}(T)$.

The following theorem is extracted from [19]. It characterizes the class $\mathcal{A}(T)$ and should be compared with Lemma 3.1 and the characterizations of $\mathcal{E}_T$ in [15, Theorem 2.1]. Remember that for each formula ϕ, $\mathrm{fv}(\phi)$ is the set of free variables of ϕ.

THEOREM 4.1. *For each theory* T *and structure* $\mathfrak{A} \in \mathcal{S}(T)$ *the following are equivalent.*

(*i*) $\mathfrak{A} \in \mathcal{A}(T)$.

(ii) *For each* $\mathfrak{A}$*-assignment* x *and pair* ϕ_1, ϕ_2 *of* $\forall_1$*-formulas such that* $T \vdash \phi_1 \vee \phi_2$, *there are* $\exists_1$*-formulas* θ_1, θ_2 *such that* $\mathrm{fv}(\theta_1) \subseteq \mathrm{fv}(\phi_1)$, $\mathrm{fv}(\theta_2) \subseteq \mathrm{fv}(\phi_2)$, $\mathfrak{A} \models \theta_1 \vee \theta_2[x]$ *and* $T \vdash \theta_1 \to \phi_1$, $T \vdash \theta_2 \to \phi_2$.

Trivially, for each theory T we have $\mathcal{A}(T) \models T \cap \forall_1$, but we do not necessarily have $\mathcal{A}(T) \models T \cap \forall_2$. In particular for number theories T we may not have $\mathcal{A}(T) \models B$. Thus, for each number theory T we put $\mathcal{A}'(T) = \mathcal{A}(T) \cap \mathcal{M}(B)$.

We easily check that $\mathcal{E}_T \subseteq \mathcal{A}'(T)$, and these classes can be distinct. For instance, let T be a complete number theory such that there is some true $\forall_1$-sentence γ with $\neg\gamma \in T$. Since T is complete, it has JEP and so $\mathfrak{N} \in \mathcal{A}'(T)$. But $\mathfrak{N} \notin \mathcal{E}_T$ for otherwise we have, for each $\mathfrak{A} \models T$; $\mathfrak{N} <_1 \mathfrak{A}$ and so $\gamma \in T$.

On first aim of this section is to extend Theorem 2.4, which is a result concerning the class $\mathcal{E}_T$, to cover the class $\mathcal{A}'(T)$. We do this by dualizing the methods of Section 2.

Theorem 2.4 is based on a certain simple set. The dual analogues of simple sets are pairs of strongly inseparable sets, so we require such a pair where the defining properties of the pair are provable in B. Analysing the construction of [14, Ex. 8–39], we have the following lemma, which should be compared with Lemma 2.1.

LEMMA 4.2. *There is a certain* $L(v)$, $R(v)$ *of* $\exists_1$*-formulas such that*

$$B \vdash (\forall v)[\neg L(v) \vee \neg R(v)]$$

and both the sentences

$$(\forall w)[(\forall v)[d(v, w) \rightarrow \neg L(v)] \rightarrow (\forall v)[d(v, w) \rightarrow v \leqq 3w \vee R(v)]]$$

$$(\forall w)[(\forall v)[d(v, w) \rightarrow \neg R(v)] \rightarrow (\forall v)[d(v, w) \rightarrow v \leqq 3w \vee L(v)]]$$

are also provable in B.

The following analogue of Theorem 2.2 is the crucial theorem.

THEOREM 4.3. *For each number theory* T, *structure* $\mathfrak{A} \in \mathscr{A}'(\mathrm{T})$, *and element* a *of* $\mathfrak{A}$, *if* $\mathfrak{A} \models \neg[L(a) \vee R(a)]$ *then* $a \in \omega$.

PROOF. Let T, $\mathfrak{A}$, a be as in the statement.

Since (by Lemma 4.2) $\mathrm{T} \vdash (\forall v)[\neg L(v) \vee \neg r(v)]$, Theorem 4.1 gives us some $l, r \in \omega$ such that

$$\mathfrak{A} \models d(a, l) \vee d(a, r)$$

and

$$\mathrm{T} \vdash (\forall v)[d(v, l) \rightarrow \neg L(v)]$$

$$\mathrm{T} \vdash (\forall v)[d(v, r) \rightarrow \neg R(v)].$$

These two sentences are $\forall_1$ and so hold in $\mathfrak{A}$, hence Lemma 4.2 gives

$$\mathfrak{A} \models (\forall v)[d(v, l) \rightarrow v \leqq 3l \vee R(v)]$$

$$\mathfrak{A} \models (\forall v)[d(v, r) \rightarrow v \leqq 3r \vee L(v)]$$

so that

$$\mathfrak{A} \models a \leqq 3t \vee L(a) \vee R\ a),$$

where $t = \max\{l, r\} \in \omega$. But $\mathfrak{A} \models \neg L(a) \wedge \neg R(a)$, and so $a \leqq 3t$, which gives the required result, $a \in \omega$.

Now let $I(x)$ be the $\exists_2$-formula

$$(\exists v)[x \leqq v \leqq 3x \wedge \neg[L(v) \vee R(v)]],$$

so this $I(x)$ is the analogue of the $I(x)$ used in Section 2. The formulas $L(v)$, $R(v)$ are chosen so that the following holds.

LEMMA 4.4. (i) *For each* $n \in \omega$, $\mathrm{B} \vdash I(n)$.

(ii) $\mathrm{P} \vdash (\forall x) I(x)$.

This lemma together with Theorem 4.3 gives us the following analogue of Theorem 2.4.

THEOREM 4.5. *There is a certain $\exists_2$-formula I containing just one free variable such that for each number theory* T *and each* $\mathfrak{A} \in \mathscr{A}'(\mathrm{T})$, $I^{\mathfrak{A}} = \omega$.

The following surprising result should be compared with Corollary 3.5. Since any model complete theory has AP, this result strengthens [0, Theorem 3.1]. It is proved in exactly the same way as [0, Theorem 3.1].

COROLLARY 4.6. *No number theory has* AP.

COROLLARY 4.7. *For each number theory* T *the only possible member of* $\mathscr{A}(\mathrm{T}) \cap \mathscr{M}(\mathrm{P} \cap \forall_3)$ *is (up to isomorphism)* $\mathfrak{N}$.

PROOF. Since $(\forall x) I(x) \in \mathrm{P} \cap \forall_3$.

Although in his thesis [3] Hirschfeld is concerned almost entirely with full number theory N, in fact most of his results hold for all number theories. (We can see this simply by verifying the provability of certain true sentences.) However, some of his results cannot be generalized by such a straightforward procedure (if at all). We will now give an example of this.

Consider [3, Theorem 2.4], this says that $\mathscr{E}_{\mathrm{T}} = \mathscr{A}(\mathrm{N}) \cap \mathscr{M}(\mathrm{N} \cap \forall_2)$. Now for each number theory T we easily see that

$$\mathscr{E}_{\mathrm{T}} \subseteq \mathscr{A}(\mathrm{T}) \cap \mathscr{M}(\mathrm{T} \cap \forall_2) \subseteq \mathscr{A}'(\mathrm{T})$$

and the example given earlier in this section, which shows that for some T $\mathscr{E}_{\mathrm{T}} \neq \mathscr{A}'(\mathrm{T})$, in fact shows that $\mathscr{E}_{\mathrm{T}} \neq \mathscr{A}(\mathrm{T}) \cap \mathscr{M}(\mathrm{T} \cap \forall_2)$ is possible. Thus the obvious generalization of Hirschfeld's theorem is false.

Let us take a closer look at Hirschfeld's proof. To do this we must go back to the effective version of the splitting theorem, i.e. Theorem 3.3.

Part (iii) of this theorem can be rephrased as follows (where M = N).

(4.1) *For each pair $X(v)$, $Y(v)$ of $\exists_1$-formulas defining r.e. sets X, Y such that $L \cap X = R \cap Y = \varnothing$, there are x, $y \in \omega$ such that*

$$\mathrm{M} \vdash (\forall v)[d(v,x) \leftrightarrow X(v) \wedge \neg R(v)]$$

$$\mathrm{M} \vdash (\forall v)[d(v,y) \leftrightarrow Y(v) \wedge \neg L(v)].$$

If we could replace 'M' in (4.1) by 'B', then we could get a completely effective version of the splitting theorem. We will show that this is not possible.

It seems likely that there is no r.e. theory M satisfying (4.1). If such an r.e. M exists, then we have a recursive procedure which produces the indexes x, y from the formula $E(v)$ (of Theorem 3.3). Looking at the proof of Theorem 3.2 we see that this is unikely.

We are grateful to Mike Yates for these, and other, remarks concerning the splitting theorem.

Let M be any $\forall_2$-axiomatizable number theory which satisfies (4.1). Clearly we could put $\mathrm{M} = \mathrm{N} \cap \forall_2$ but we would prefer, if possible, to use a much smaller theory.

The following is a generalization of Hirschfeld's theorem.

THEOREM 4.8. *For each number theory* T, $\mathscr{A}(\mathrm{T}) \cap \mathscr{M}(\mathrm{M}) \subseteq \mathscr{E}$, *and if* $\mathrm{M} \subseteq \mathrm{T}$ *then* $\mathscr{A}(\mathrm{T}) \cap \mathscr{M}(\mathrm{M}) = \mathscr{E}_{\mathrm{T}}$.

PROOF. The proofs of both statements are similar, so we will prove only the second one.

Suppose $\mathrm{M} \subseteq \mathrm{T}$. Since M is $\forall_2$-axiomatizable we have $\mathscr{E}_{\mathrm{T}} \subseteq \mathscr{A}(\mathrm{T}) \cap \mathscr{M}(\mathrm{M})$, thus we must show the reverse inclusion.

Let $\mathfrak{A} \in \mathscr{A}(\mathrm{T})$, $\mathfrak{A} \models \mathrm{M}$. We will verify [15, Theorem 2.1 (iii)] and hence get $\mathfrak{A} \in \mathscr{E}_{\mathrm{T}}$. Since T has recursive pairing functions, it is sufficient to verify this condition for $\forall_1$-formulas containing just one free variable.

Let $\mathfrak{A} \models \phi(a)$ for some $\forall_1$-formula $\phi(v)$ and element a of $\mathfrak{A}$. Let $L(v)$, $R(v)$ be the $\exists_1$-formulas given by Theorem 3.3 applied to the formula $E(v) = \neg\phi(v)$.

Since $\mathfrak{A} \in \mathscr{A}(\mathrm{T})$, Theorem 3.3 (ii) and Theorem 4.1 gives us $\exists_1$-formulas $X(v)$, $Y(v)$ such that

$$\mathfrak{A} \models X(a) \vee Y(a) \tag{4.2}$$

and

$$\begin{gathered} \mathrm{T} \vdash (\forall v)[X(v) \rightarrow \neg L(v)] \\ \mathrm{T} \vdash (\forall v)[Y(v) \rightarrow \neg R(v)]. \end{gathered} \tag{4.3}$$

We check that these give $L \cap X = R \cap Y = \varnothing$ (where L, R, X, Y are the corresponding real life r.e. sets) so that (4.1) gives us some $x, y \in \omega$ such that

$$\begin{gathered} \mathrm{M} \vdash (\forall v)[d(v, x) \leftrightarrow X(v) \wedge \neg R(v)] \\ \mathrm{M} \vdash (\forall v)[d(v, y) \leftrightarrow X(v) \wedge \neg L(v)]. \end{gathered} \tag{4.4}$$

Now, from (4.2), we can assume that $\mathfrak{A} \models X(a)$. Also (since $\mathfrak{A} \models \phi(a)$) Theorem 3.3(i) gives us $\mathfrak{A} \models \neg R(a)$, so that (4.4) gives

$$\mathfrak{A} \models \theta(a),$$

where $\theta(v) = d(v, x)$. We now use, for the first time, $\mathrm{M} \subseteq \mathrm{T}$ which, with (4.4), (4.3) and Theorem 3.3 (i), gives

$$\mathrm{T} \vdash (\forall v)[\theta(v) \rightarrow \phi(v)].$$

Thus we have verified that $\mathfrak{A} \in \mathscr{E}_{\mathrm{T}}$.

Corollary 4.9. $\mathrm{M} \neq \mathrm{B}$.

Proof. If $\mathrm{M} = \mathrm{B}$ then, for each number theory T,

$$\mathscr{A}'(\mathrm{T}) = \mathscr{A}(\mathrm{T}) \cap \mathscr{M}(\mathrm{M}) = \mathscr{E}_{\mathrm{T}},$$

which is false.

5. Cores of number structures

We say a structure $\mathfrak{A}$ is rigidly contained in a structure $\mathfrak{B}$ if there is exactly one embedding of $\mathfrak{A}$ into $\mathfrak{B}$. For instance, the standard number structure is rigidly contained in every number structure. A structure $\mathfrak{A}$ is a rigid part of a theory T if $\mathfrak{A}$ is rigidly contained in every model of T.

These definitions immediately give us the following lemma.

Lemma 5.1. *If $\mathfrak{A}, \mathfrak{B}$ are models of a theory* T *and both are a rigid part of* T *then $\mathfrak{A} \cong \mathfrak{B}$ and this isomorphism is unique.*

Of course, in general a rigid part of a theory is not necessarily a model of the theory.

Let $\mathfrak{A} \in \mathscr{S}(\mathrm{T})$. We say an element a of $\mathfrak{A}$ is $\exists_1$-definable over T if there is an $\exists_1$-formula $\theta(v)$ such that

$$\mathfrak{A} \models \theta(a), \quad \mathrm{T} \vdash (\exists! v)\theta(v).$$

A proof of the following can be obtained from [22, Theorem 2.1 and Corollary 2.4].

Theorem 5.2. *For each structure $\mathfrak{A}$ and theory* T *the following are equivalent.*

(i) *$\mathfrak{A}$ is a rigid part of* T.

(ii) *$\mathfrak{A}$ is a model of each $\forall_1$-sentence which is consistent with* T, *and each element of $\mathfrak{A}$ is $\exists_1$-definable over* T.

If a theory T has a maximal rigid part then we call this rigid part the hard core of T. For instance, $\mathfrak{N}$ is the hard core of B. Notice that this is an example of a theory without JEP but having a hard core. This shows that the converse of the next theorem is false. This theorem follows easily from Theorem 5.2.

THEOREM 5.3. *Let* T *be a theory with* JEP. *Then* T *has a hard core. For each model* $\mathfrak{A}$ *of* T *the set* $E(\mathfrak{A})$ *of elements of* $\mathcal{A}$ *which are* $\exists_1$*-definable over* T *forms the hard core of* T.

For each number structure $\mathfrak{A}$ let

$$K(\mathfrak{A}) = \bigcap\{\mathfrak{B}\colon\ \mathfrak{B} <_1 \mathfrak{A}\}.$$

We call $K(\mathfrak{A})$ the core of $\mathfrak{A}$.

THEOREM 5.4. *For each number structure* $\mathfrak{A}$, $K(\mathfrak{A}) <_1 \mathfrak{A}$ *and so* $K(\mathfrak{A})$ *is also a number structure.*

PROOF. By Corollary 1.3.

COROLLARY 5.5. *For each number structure* $\mathfrak{A}$, $K^2(\mathfrak{A}) = K(\mathfrak{A})$.

THEOREM 5.6. *For each number structure* $\mathfrak{A}$ *and element* a *of* $\mathfrak{A}$ *the following are equivalent.*

(i) *a is an element of* $K(\mathfrak{A})$.

(ii) *a is* $\exists_1$*-definable over* $\mathrm{Th}(\mathfrak{A})$.

PROOF. Let $E(\mathfrak{A})$ be the substructure of elements of $\mathfrak{A}$ which are $\exists_1$-definable over $\mathrm{Th}(\mathfrak{A})$. First we show that $E(\mathfrak{A}) <_1 \mathfrak{A}$.

Consider any $\exists_1$-formula $\phi(w_1, \cdots, w_s)$ and elements $a_1, \cdots, a_s$ of $E(\mathfrak{A})$ such that $\mathfrak{A} \models \phi(a_1, \cdots, a_s)$. Let

$$\phi(w_1, \cdots, w_s) = (\exists v_1, \cdots, v_r)\theta(v_1, \cdots, v_r, w_1, \cdots, w_s),$$

where θ is quantifier-free, and let $\mu(v_1, \cdots, w_s)$ be the corresponding $\exists_1$-formula given by Lemma 1.1.

For each $1 \leqq i \leqq s$ let $\theta_i(v)$ be some $\exists_1$-formula such that

$$\mathfrak{A} \models \theta_i(a_i),\ \ \mathfrak{A} \models (\exists!\, v)\theta_i(v),$$

and let $\psi(v_1, \cdots, v_r)$ be the $\exists_1$-formula

$$(\exists w_1, \cdots, w_s)[\theta_1(w_1) \wedge \cdots \wedge \theta_s(w_s) \wedge \mu(v_1, \cdots, v_r, w_1, \cdots, w_s)].$$

We see that $\mathfrak{A} \models (\exists!\, v_1, \cdots, v_r)\psi(v_1, \cdots, v_r)$ so there are elements $b_1, \cdots, b_r$ of $E(\mathfrak{A})$ with $\mathfrak{A} \models \psi(b_1, \cdots, b_r)$. This gives $\mathfrak{A} \models \mu(b_1, \cdots, b_r, a_1, \cdots, a_s)$ and so $\mathfrak{A} \models \theta(b_1, \cdots, b_r, a_1, \cdots, a_s)$.

But θ is quantifier-free, hence $E(\mathfrak{A})\models\theta(b_1,\cdots,b_r,a_1,\cdots,a_s)$, which gives $E(\mathfrak{A})\models\phi(a_1,\cdots,a_s)$, as required.

This shows that $E(\mathfrak{A})<_1\mathfrak{A}$ so that $K(\mathfrak{A})\subseteq E(\mathfrak{A})$. It remains to show that $E(\mathfrak{A})\subseteq K(\mathfrak{A})$.

Consider any element a of $E(\mathfrak{A})$ and let $\theta(v)$ be an $\exists_1$-formula such that

$$\mathfrak{A}\models\theta(a),\quad \mathfrak{A}\models(\exists!\,v)\theta(v).$$

Since $K(\mathfrak{A})<_1\mathfrak{A}$ we have $K(\mathfrak{A})\models(\exists!\,v)\theta(v)$, and so a is an element of $K(\mathfrak{A})$, as required.

This completes the proof.

Corollary 5.7. *For each number structure* $\mathfrak{A}$, $K(\mathfrak{A})$ *is the hard core of* Th($\mathfrak{A}$).

In the next corollary we write $\mathfrak{A}\equiv_1\mathfrak{B}$ to indicate $\mathfrak{A}\Rrightarrow(\exists_1)\mathfrak{B}$ and $\mathfrak{B}\Rrightarrow(\exists_1)\mathfrak{A}$.

Corollary 5.8. *For each number structures* $\mathfrak{A}$, $\mathfrak{B}$,

$$\mathfrak{A}\equiv_1\mathfrak{B}\Leftrightarrow K(\mathfrak{A})\overset{!}{\cong}K(\mathfrak{B}).$$

This isomorphism (when it exists) is unique.

We have seen that standard part of certain number structures can be defined within the structure. We now see that the core of these structures can be defined.

Let $K(v)$ be the $\exists_2$-formula

$$(\exists x)[I(x)\wedge d(v,x)\wedge(\forall v')[d(v',x)\rightarrow v'=v]]$$

(where $I(x)$ is the formula of Theorem 4.5). Theorem 5.6 gives the following theorem.

Theorem 5.9. *For each number theory* T *and structure* $\mathfrak{A}\in\mathscr{A}'(\mathrm{T})$, $K^{\mathfrak{A}}=K(\mathfrak{A})$.

We now look at the f-companion of complete number theories. The following theorem is essentially due to Hirschfeld [3, Theorem 6.7]. It should be compared with [22, Theorem 2.5(f)].

Theorem 5.10. *For each number structure* $\mathfrak{A}$,

$$\mathrm{Th}(\mathfrak{A})^f=\mathrm{Th}(K(\mathfrak{A}))$$

and (up to isomorphism) $K(\mathfrak{A})$ *is the unique member of* $\mathscr{F}_{\mathrm{T}}$*(where* $\mathrm{T}=\mathrm{Th}(\mathfrak{A})$*).*

PROOF. Let $\mathfrak{B}$ be any structure such that $K(\mathfrak{A}) \subseteq \mathfrak{B}$ and $K(\mathfrak{A}) \equiv \mathfrak{B}$. Consider any formula $\phi(v_1, \cdots, v_r)$ and elemements $a_1, \cdots, a_r$ of $K(\mathfrak{A})$ such that $K(\mathfrak{A}) \models \phi(a_1, \cdots, a_r)$.

By Theorem 5.6 we have $\exists_1$-formulas $\theta_1(v), \cdots, \theta_r(v)$ such that for each $1 \leqq i \leqq r$

$$K(\mathfrak{A}) \models \theta_i(a_i), \; K(\mathfrak{A}) \models (\exists ! v)\theta_i(v).$$

Hence we have

$$K(\mathfrak{A}) \models (\forall v_1, \cdots, v_r)[\wedge \{\theta_i(v_i) \colon 1 \leqq i \leqq r\} \to \phi(v_1, \cdots, v_r)].$$

Now $A(\mathfrak{A}) \subseteq \mathfrak{B}$, $K(\mathfrak{A}) \equiv \mathfrak{B}$ and each θ_i is $\exists_1$, so we can transfer to $\mathfrak{B}$, and so obtain $\mathfrak{B} \models \phi(a_1, \cdots, a_r)$.

This shows that $K(\mathfrak{A}) < \mathfrak{B}$, and so $K(\mathfrak{A})$ is a completing model of $\mathrm{Th}(K(\mathfrak{A}))$. It follows that $\mathrm{Th}(K(\mathfrak{A}))$ is f-complete, hence, since $\mathrm{Th}(\mathfrak{A}) \cap \forall_1 = \mathrm{Th}(K(\mathfrak{A})) \cap \forall_1$, $\mathrm{Th}(K(\mathfrak{A}))$ is the f-companion of $\mathrm{Th}(\mathfrak{A})$ and $K(\mathfrak{A}) \in \mathscr{F}_{\mathrm{T}}$(where $\mathrm{T} = \mathrm{Th}(\mathfrak{A})$).

Now let $\mathfrak{B}$ be any member of $\mathscr{F}_{\mathrm{T}}$ (i.e. a completing model of $\mathrm{Th}(K(\mathfrak{A}))$. Since $K(\mathfrak{A}) \models (\forall v)K(v)$, $K(\mathfrak{A}) \equiv \mathfrak{B}$, and $\mathfrak{B} \in \mathscr{E}$, we have

$$\mathfrak{B} = K^{\mathfrak{B}} = K(\mathfrak{B}) \cong K^2(\mathfrak{A}) = K(\mathfrak{A}),$$

which shows the required uniqueness of $\mathfrak{A}$.

COROLLARY 5.11. *The theory* F *is a finite extension of* E, *being axiomatized over* E *by* $(\forall v)K(v)$.

Finally, since every non-standard member of $\mathscr{E}$ satisfies $\neg(\forall v)I(x)$, Corollary 5.7 gives the following theorem.

THEOREM 5.12. *For each complete number theory* T, *either the hard core of* T *is* $\mathfrak{N}$ *or the hard core is not a model of* $\mathrm{P} \cap \forall_3$.

6. Further remarks and comments

It this last section we suggest one or two further lines of research.

1. Much of the tedium of [0] and this paper arises from the need to verify provability in P. Is there any reasonable way out of this? Thus, is there any test which for some sentences $\sigma \in \mathrm{N} \cap \forall_2$ will show $\mathrm{P} \vdash \sigma$? Clearly there is no such test which works for all $\sigma \in \mathrm{N} \cap \forall_2$.

2. Describe the set of sentences $\sigma \in \mathrm{P}$ such that for each $\forall_2$-axiomatizable number theory T, $\sigma \notin \mathrm{T}$.

3. The method of proof of Theorem 3.4 is similar to that used in [23]. What is the relevance of Theorem 3.4 to the result of [23]?

4. Find a number theory T such that the three classes $\mathcal{S}'(N)$, $\mathcal{I}(T)$, $\mathcal{S}'(T)$ are distinct.

5. The equivalence relation $\equiv_1$ partitions the class $\mathcal{M}(B)$ into several equivalence classes, which we call nodes. The relation $\Rrightarrow(\exists_1)$ induces a partial ordering on the set of nodes. Theorem 3.11 shows that this partial order is a tree. This tree was introduced in [21, Theorem 6] where further properties of it can be found. There are many unsolved problems concerning this tree, for instance see [21, Problem 7].

6. Theorem 4.5 apparently extends Theorem 2.4. Justify this by finding a number theory T and structure $\mathfrak{A} \in \mathcal{A}'(T)$ such that $\mathfrak{A} \notin \mathscr{E}$. However see 8.

7. Corollary 4.6 ought to have a more direct proof. What model theoretic properties of an $\forall_2$-axiomatizable theory B imply that no extension of B has AP?

8. The situation concerning the theory M and Theorem 4.8 is unsatisfactory. Perhaps some insight can be gained by attempting to prove the following.

(i) If M satisfies (4.1) then $N \cap \forall_2 \subseteq M$.

(ii) For each number theory T, $\mathcal{A}'(T) \subseteq \mathscr{E}$.

Notice that a proof of (ii) would show that most of Section 4 is pointless.

9. Can Corollary 5.8 be improved to

$$\mathfrak{A} \Rrightarrow (\exists_1) \Leftrightarrow K(\mathfrak{A}) \text{ is embeddable in } K(\mathrm{B})$$

where, of course, $\mathfrak{A}$ and $\mathfrak{B}$ are number structures? [Yes, H.S.]

10. Use Theorem 5.10 to give a description of $\mathcal{F}_T$ for arbitrary number theories T.

REFERENCES

0. D. C. Goldrei, A. Macintyre and H. Simmons, *The forcing companions of number theories*, Israel J. Math. **14** (1973), 317–337.

3. J. Hirschfeld, *Existentially complete and generic structures in arithmetic*, Doctoral dissertation, University of Yale, 1972.

7. Now published in Colloquium Math. **28** (1973), 165–180.

13. Now published in Proceedings of the fourth International Congress for Logic, Methology and Philosophy of Science, Bucharest, 1971, Edited by P. Suppes et al, North Holland, Amsterdam, 1973, pp. 137–154.

16. Now published in Math. Scand. **31** (1972), 257–261.

17. Now published in Compositio Math **28** (1974), 113–142.

18. W. H. Wheeler, *Algebraically closed division rings, forcing and the analytic hierarchy*, Doctoral dissertation, University of Yale, 1972.

19. P. D. Bacsich and D. Rowlands-Hughs, *Syntactic characterisations of amalgamation, convexity and related properties*, J. Symbolic Logic **39** (1974), 433–451.

20. J. P. Cleave, *Some properties of recursively inseparable sets*, Z. Math. Logik Grundlagen Math. **16** (1970), 187–200.

21. D. Jensen and A. Ehrenfeucht, *Some problems in elementary arithmetics*, to appear in Fund. Math.

22. D. W. Kueker, *Core structures for theories*, University of Maryland technical report TR 73–21.

23. J. Myhill, *An absolutely independent set of* Σ_1^0*-sentences*, Z. Math. Logik Grundlagen Math. **18** (1972), 107–109.

24. H. Simmons, *Le nombre de structures générique d'une théorie*, C. R. Acad. Sci. Paris Sér. A **277** (1973), 487–489.

UNIVERSITY OF YALE
AND
UNIVERSITY OF ABERDEEN